Exercise 66

Two curves are **orthogonal** if their tangent lines are perpendicular at each point of intersection. Show that the given families of curves are **orthogonal trajectories** of each other; that is, every curve in one family is orthogonal to every curve in the other family. Sketch both families of curves on the same axes.

$$x^2 + y^2 = ax, \quad x^2 + y^2 = by$$

Solution

The points of intersection are found by solving the system of equations for x and y.

$$\begin{cases} x^2 + y^2 = ax \\ x^2 + y^2 = by \end{cases}$$

Subtract the respective sides of each equation to eliminate the squared terms.

$$0 = ax - by$$

Differentiate both sides of the given equations with respect to x.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(ax) \qquad \frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(by)$$

Use the chain rule to differentiate $y = y(x)$.

$$2x + 2y\frac{dy}{dx} = a \qquad 2x + 2y\frac{dy}{dx} = b\frac{dy}{dx}$$

Solve each equation for dy/dx.

$$2y\frac{dy}{dx} = a - 2x \qquad (2y - b)\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{a - 2x}{2y} \qquad \frac{dy}{dx} = -\frac{2x}{2y - b}$$

Check to see that these slopes are negative reciprocals.

$$-\frac{2y}{a - 2x} \stackrel{?}{=} -\frac{2x}{2y - b}$$

$$\frac{y}{a - 2x} \stackrel{?}{=} \frac{x}{2y - b}$$

$$y(2y - b) \stackrel{?}{=} x(a - 2x)$$

$$2y^2 - by \stackrel{?}{=} ax - 2x^2$$

$$2x^2 + 2y^2 \stackrel{?}{=} ax + by$$

Continue the simplification.

$$2(x^2 + y^2) \stackrel{?}{=} ax + by$$

$$2(by) \stackrel{?}{=} ax + by$$

$$0 \stackrel{?}{=} ax - by$$

$$0 = 0$$

The slopes are negative reciprocals at the points of intersection; therefore, the familes of curves defined by $x^2 + y^2 = ax$ and $x^2 + y^2 = by$ are orthogonal trajectories. Observe that at all points of intersection the tangent lines to the members of each family of curves are orthogonal.